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RADIATION CHARACTERISTICS OF CIRCULAR AND SEMICIRCULAR SURFACE SOURCES

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ABSTRACT: In this report, a theoretical study of the radiation characteristics for both semicircular and circular surface sources has been made, when these sources radiate uniformly over their surfaces and obey Lambert's Cosine Law. This study includes a presentation of the history of the development of such formulae from the time of Lambert in 1760 up to the present. Very general equations for both types of sources have been derived giving the total flux falling on an elementary receiving area, when this elementary receiving area has arbitrary coordinates and a surface normal with arbitrary direction cosines. The classical method of surface integration, introduced by Lambert, has been used in each case. These equations are very general in form, and it is shown how each of the equations found in the literature becomes a special case of these more general equations, in which the coordinates of the elementary receiving area or the direction cosines of its surface normal have particular values. After a simple translation and rotation of coordinates, these equations can be used for the equally important problem of calculating the total flux falling on the elementary receiving area in the presence of an array of such sources having arbitrary coordinates and surface normals with arbitrary direction cosines. A numerical example of radiation field calculations for each of these sources is given.

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This report represents a theoretical study of the radiation characteristics of Circular and Semicircular Surface Sources. It is intended for information only. It is published under Task FR-1-54.

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RADIATION CHARACTERISTICS OF CIRCULAR AND SEMICIRCULAR SURFACE SOURCES

I. INTRODUCTION

1. In a previous report, reference (a), we discussed the radiation characteristics of rectangular surface sources when these sources radiate uniformly over their surfaces and obey Lambert's Cosine Law. As stated, these calculations are only valid for radiant energy when the wave length is small compared to the elementary source areas and is of random phase across the surface of the rectangle. In the case of particles, the particle size must also be small compared to these elementary source areas and not interact if these surface integrations are to be of value. This report presented two very general formulae for calculating the total flux falling on an elementary receiving area from the rectangular source when the elementary receiving area had arbitrary coordinates and a surface normal with arbitrary direction cosines. A brief history of the development of this problem was also given, and it was shown how all the formulae found in the literature, except one, were special cases of these more general formulae in which either the coordinates of the elementary receiving area or the direction cosines of its surface normal had particular values. Some examples of the form of the radiation field in particular cases were also given for which the elementary receiving area was confined to a given line or plane, and both graphs and detailed plots of isophotopic lines in given planes were presented.

2. In this report, we should like to extend these considerations by presenting similar material for both the semicircular and circular surface sources. These sources also are quite frequent in occurrence and may, like the rectangular source, be used in building up the radiation fields of surfaces of more complex geometry, and of surfaces which do not radiate uniformly over their surfaces and do not obey Lambert's Cosine Law.

II. HISTORY

3. The problem of the circular radiating surface source is quite an old one. If one assumes that this source radiates uniformly over its surface and obeys Lambert's Cosine Law, (reference (b)), then J. H. Lambert in 1760 gave a formula, (reference (b) p. 80), for the flux density on an elementary

receiving area located on the surface normal to the circular source through its center, and having its surface normal perpendicular to the surface of the circular source as suggested by Figure 1. This formula can be derived by surface integration and has been presented by many later writers, (reference (c)). Lambert also considered the more complicated case in which the elementary receiving area was moved away from the axial position as suggested by Figure 2, its surface normal still remaining parallel to the axis through the center of the circular surface, (reference (b)). This formula is a little more complicated and has been presented by many later writers, (reference (d)). In the literature it may be of interest to note that there is a paper on the latter problem, (reference (e)), which points out that it cannot be solved rigorously by surface integration methods. Other methods are then discussed involving potential theory and various approximations which are not necessary. The formulae for the problems suggested by Figures 1 and 2 are the ones appearing most often in the literature, and other formulae for the circular source become increasingly difficult to find.

4. While discussing the properties of diffuse reflectors, C. Hillebrand, (reference (f)), presented a formula for the total flux falling on an elementary receiving area located on the axis, but having a surface normal with arbitrary direction cosines as suggested by Figure 3. This formula seems to be the first for which the direction cosines of the surface normal of the elementary receiving area were not restricted to a fixed direction.

5. P. D. Foote, (reference (d)), in his paper in 1915 presented an equation for the circular source, which gave the flux density on a small receiving area located on a normal to the circular surface through its circumference and having a surface normal perpendicular to the circular surface as suggested by Figure 4. Actually, this formula is but a slight modification of Lambert's second formula for Figure 2.

6. One of the most general papers on the circular source is that published by V. Fock, (reference (g)), in 1924. In this paper, the author further extends these ideas by presenting three formulae for the flux density on the elementary receiving area when it has arbitrary coordinates and is oriented in the three directions of the coordinates as suggested by Figure 5. In deriving these three equations, the author introduces a "vector potential of illumination" which makes it possible, from Stoke's Law, to reduce surface integrations to line integrations. Knowing the components of this "vector potential of illumination", final values for the flux density on the elementary receiving area having arbitrary coordinates and oriented with its surface normal

along the directions of the axes can be evaluated by calculating the components of the curl of this vector potential of illumination. This paper represents a further generalization of the problem beyond those suggested by the previous figures. However, it has even further implications. These three equations for the flux density on the elementary receiving area, having arbitrary coordinates and oriented with its surface normal in the directions of the coordinates as suggested by Figure 5, are looked upon as components of a "light vector". If these components of the "light vector" are substituted in the proper vector equation, it becomes possible to write an expression for the "light vector" for any position and orientation.

7. J. W. T. Walsh in his book on Photometry, (reference (h)), gives an equation for the circular source when the elementary receiving area is oriented with its surface normal along the Y axis as in Figure 6. This equation should be a special case of one of Fock's equations but unfortunately it has been written down incorrectly. This has been acknowledged by the author and is corrected in the recent edition of the book.

8. Very recently, interest in these problems has been revived by the detailed papers of Bethe, (reference (i)). Bethe returns to the method of surface integration suggested by Lambert and discusses surface sources of various types. In this paper, he presents an equation for the flux density from a circular surface source when the elementary receiving area is limited to the X_0 axis and has a surface normal with arbitrary orientations. The circular source itself, however, is limited to the (Y,Z) plane and has its center along the Y axis as suggested in Figure 7. This equation is more general in form than those of Fock. Bethe then presents Hillebrand's equation for Figure 3 as a special case of his equation.

9. Very little was found in the literature on the subject of semicircular surface sources. One of the few people to discuss some of the simpler aspects of this problem was U. Bordoni, (reference (c)), who also, as suggested earlier, worked with circular surface sources. In his paper, he published two formulae for the flux density on an elementary receiving area when the semicircular source and elementary receiving area were arranged as in Figures 8 and 9. Figure 8 obviously is a trivial example of Figure 1 for the circular source involving simply a factor of two when comparing the two formulae. Figure 9, however, is a distinctly different and more complicated problem than the corresponding one for the circular source.

10. Another problem for the semicircular source which has been solved is that indicated in Figure 10. This problem is a more general case of that already cited in Figure 9 and has been discussed by Walsh in his book on Photometry, (reference (h)). The formula for this problem becomes much more complicated. Bethe, (reference (i)), in his paper also discusses semicircular sources very briefly and again gives Bordoni's formula for Figure 9. These three formulae represent the only contributions which were found on this subject.

11. Recently we became interested in the more general problem of surface sources of semicircular and circular shapes radiating flux of an arbitrary type to an elementary receiving area having arbitrary coordinates and a surface normal with arbitrary direction cosines. These problems, like the corresponding one for the rectangular source discussed in the previous report, arose in connection with methods for calculating the total flux falling on an elementary receiving area from sources not only of these simple shapes but of more complex geometries. Considerations of this same type also arise for surface sources which do not radiate uniformly over their surfaces or obey Lambert's Cosine Law. One method of solving these more complex problems is to divide these surface areas into smaller areas of simpler known geometric forms (rectangles, circles, semicircles) which do approximately satisfy these conditions (radiate uniformly over their surfaces and obey Lambert's Cosine Law), and then calculate the separate contributions of each. The total flux from the complete source is then the sum of the contributions of each of these smaller sources. In this connection, it became necessary to know the total flux falling on an elementary receiving area from circular and semicircular sources when either the elementary receiving area or the source had arbitrary coordinates and a surface normal with arbitrary direction cosines. The previously mentioned literature search did not give a direct answer to these problems. The closest solution to the problem for the circular source was the work of Fock, in which he gave the components of the "light vector" for an arbitrary position in space, and suggested that a more general vector equation might be written giving the "light vector" for any position and orientation in space. This equation, however, was never written because of the author's interest in simpler aspects of the problem. Such an equation, if it had been written, would have represented the only attempt at such a general equation for the circular source found in the literature. No similar equation or suggestion of an equation for the semicircular source was found.

12. As a result of this literature search, we felt it might be worthwhile to solve these two problems independently to verify the somewhat limited results of others, and perhaps to add new and simpler formulae. As with the rectangular source, we felt, in addition, that there was also a need to bring together all these apparently very different formulae for both the semicircular and circular sources, since they are recorded in many journals which are both old and difficult to obtain, and to point out where they are similar, where they deviate from each other, and perhaps also in a few cases whether or not they are correct. The solutions to these two problems will do this, for all the formulae found in the literature for these two sources were, as we shall see later, special cases of these formulae in which either the coordinates of the elementary receiving area or the direction cosines of its surface normal had particular values.

13. In solving these two problems, we were, as in the previous case for the rectangular source, confronted with choosing a method of calculation. Should it be the older method of surface integration suggested by Lambert or the more recent vector method which has not received such wide acceptance? The somewhat limited results which we did find on these two general problems were found using the vector method. However, the concepts of a "light Vector", a "vector potential of illumination", as well as the various terminologies used in describing them have not been generally accepted by workers in the field of photometry. We thought, therefore, in this paper that we would return to first principles and use the method of surface integration, since most of the simpler problems were solved in this way, and a few of the older writers as well as more recent ones have been stressing its limitations in solving such problems.

III. GENERAL EQUATION FOR RADIATION FIELD OF THE SEMICIRCULAR SURFACE SOURCE

14. To show in more detail the first of these proposed problems, it will be necessary to refer to Figure 11. At the origin of the coordinate system, we have placed a circular surface source whose center coincides with the origin of the coordinates and whose surface is limited to the (Y, Z) plane. This circular source has been divided into shaded and unshaded semicircular areas, and we will be concerned first with these semicircular areas. We will assume that both these semicircular areas act as sources of flux of some type and will not be concerned with the mechanism by which this flux arises. For our purposes, the

form which the flux takes is unimportant, for it may be a wave disturbance or consist of a flow of particles. If the flux is in the form of radiant energy, we shall not be concerned with its frequency distribution or its plane of polarization, for these properties become important only when the response of a given receiver is being considered, and this will be outside the scope of the present report. At the arbitrary point (X_0, Y_0, Z_0), we have indicated an elementary receiving area ds', whose dimensions are so small that the total flux falling on its surface from either of the semicircular sources is at all times directly proportional to its area. We shall be interested in calculating the total flux falling on this elementary receiving area. This elementary receiving area has a surface normal whose direction cosines (α, β, γ) are completely arbitrary. We will assume that these semicircular sources both radiate uniformly over their surfaces, and that they obey Lambert's Cosine Law, (reference (b)), and hence, are "perfect diffuse emitters". This, of course, represents an approximation, for many sources do not obey this law, (reference (j)). There are, however, several sources which do obey this law within a few percent (e.g., magnesium oxide, finely ground glass surfaces, thermal radiation from molten silver and unpolished platinum), particularly if the surface is somewhat rough, and if the range in angle of emergence of the flux falling on the elementary receiving area ds' is not excessive. We will, therefore, assume that Lambert's Cosine Law is valid for both the semicircular and circular sources to be discussed here. In performing these calculations, it will also be assumed that the intervening space between these sources and the elementary receiving area ds' is homogenous and does not scatter or absorb the flux on its passage through this space. The contributions of multiple reflections, multiple absorptions, and reemissions between the sources and the elementary receiving area to the total flux falling on the elementary receiving area will always be assumed to be so small that their contributions are negligible.

15. In the shaded semicircular area in Figure 11, a surface element of area ds, having coordinates (0, h sin φ, h cos φ), has been indicated. This area element has a surface normal N forming an angle θ with the line of length a joining ds and ds', and the surface normal N' to the elementary receiving area ds' forms an angle θ' with this same line. The total flux which this area element ds radiates to the elementary receiving area ds' can then be written

$$d^2F = \frac{B\,ds \cos\theta \; ds' \cos\theta'}{a^2} \qquad (1)$$

B being the total energy ** radiated by each unit area of the source per second per unit solid angle. Since we have assumed that these sources are "perfect diffuse emitters", it follows that B will be independent of θ. This equation is one of the old and universally accepted laws of photometry, (reference (b) p. 76), and is commonly referred to as Lambert's Photometric Law. It serves as the starting point for the calculations of this report as has been the case for many other papers on these subjects.

16. To find the total flux falling on the elementary receiving area ds' from the shaded semicircular area in Figure 11, equation 1 must now be integrated over the surface of the semicircular source. To do this, it is necessary to express ds, cos θ, cos θ', and a in terms of quantities suggested by this figure. The area element ds, when expressed in terms of plane polar coordinates in the plane of the semicircular source, can be replaced by h.dh.dφ. From the figure, it is quite easy to see that cos θ is equal to

$$\cos \theta = \frac{X_o}{a} \tag{2}$$

To fix the magnitude of cos θ', we use the fact that the angle between two lines whose direction cosines are known is given by

$$\cos \theta' = \alpha \cos(a,X) + \beta \cos(a,Y) + \gamma \cos(a,Z) \tag{3}$$

where α, β, and γ are the direction cosines of N'.*** The direction cosines of the line a are quite readily determined from

---

** By total energy we mean all the energy entering within the unit solid angle. Thus if the flux is in the form of particles, it must be the total energy contained by the particles which enter within this unit solid angle. If the radiation is periodic, it must be integrated over all frequencies; if it is transverse, it must be integrated over all planes of polarization.

***It perhaps should be noted that the directions cosines of N' have been chosen positive in the same sense in which a is positive so that N' will always be the surface normal to ds' not exposed to the incident flux. This same convention has been used in all the figures submitted in this report.

the figure. They are

$$\cos(a,X)=\frac{X_0}{a},\ \cos(a,Y)=\frac{Y_0-h\sin\varphi}{a},\ \cos(a,Z)=\frac{Z_0-h\cos\varphi}{a} \tag{4}$$

The magnitude of a is clearly given by

$$a=\sqrt{X_0^2+Y_0^2+Z_0^2+h^2-2Y_0h\sin\varphi-2Z_0h\cos\varphi} \tag{5}$$

If we substitute now all these quantities for ds, cos θ, cos θ', and a in equation 1, the integral equation which must be solved to determine the total flux falling on the elementary receiving area from the shaded semicircular area in Figure 11 must then be the following:

$$dF=Bds'X_0\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^R\frac{(\alpha X_0+\beta Y_0+\gamma Z_0-\beta h\sin\varphi-\gamma h\cos\varphi)\,h\,dh\,d\varphi}{(X_0^2+Y_0^2+Z_0^2+h^2-2Y_0h\sin\varphi-2Z_0h\cos\varphi)^2} \tag{6}$$

R being the radius of the circular disk. This equation unfortunately is not very easy to integrate. Even after the integration has been carried out, there is some question as to which is the "physically significant solution", for actually the final solution contains a multiplicity of possible values.

17. To determine which is the "physically significant solution", a much simpler problem indicated in Figure 12 was first solved. All we have done here is to limit the elementary receiving area in the more general problem in Figure 11 to the X_0 axis. Without going into detail then, all we need to do to obtain the integral equation for this particular problem is to set $Y_0 = Z_0 = 0$ in equation 6. The resulting integral equation which we must solve for this problem is then

$$dF=Bds'X_0\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^R\frac{\alpha X_0-\beta h\sin\varphi-\gamma h\cos\varphi}{(X_0^2+h^2)^2}\,h\,dh\,d\varphi. \tag{7}$$

This equation can now be integrated directly over both the variables φ and h. Integrating first over φ, we find that

$$dF = B\,ds'X_0 \left\{ \pi \alpha X_0 \int_0^R \frac{h\,dh}{(X_0^2 + h^2)^2} - 2\gamma \int_0^R \frac{h^2\,dh}{(X_0^2 + h^2)^2} \right\}. \tag{8}$$

These integrals are both of well known forms, so that dF for the shaded semicircular area in Figure 12 will be found to be

$$dF = B\,ds' \left\{ \frac{\pi \alpha R^2}{2(X_0^2 + R^2)} + \gamma \left[\frac{X_0 R}{X_0^2 + R^2} - \tan^{-1} \frac{R}{X_0} \right] \right\} \tag{9}$$

There is an arctangent term in this equation, so that multiple values for dF are possible depending upon the choice of the angle corresponding to this term. To determine which is the "physically significant value", we assume that $X_0 >> R$ in this equation and then make a power series development of the arctangent term.

$$dF = \frac{B\,ds'\alpha}{X_0^2} \left(\frac{\pi R^2}{2} \right) \tag{10}$$

which is, of course, what we expect from Lambert's Photometric Law in equation 1. If this is the physically significant value for large values of X_0 as compared with R, then for positive values of X_0 in Figure 12 of the order of magnitude of R or less, the arctangent term in equation 9 must be restricted to values in the first quadrant to obtain the physically significant solution.

18. If one solves the corresponding problem for the contribution of the unshaded semicircular area in Figure 12 to the total flux

falling on ds', one obtains the same integral indicated in equation 7, but the limits on ϕ are now from $\pi/2$ to $3/2\pi$. This integral can also be evaluated without further difficulty with the result that dF for the unshaded semicircular area in Figure 12 must be

$$dF = B\,ds'\left\{\frac{\pi\alpha R^2}{2(X_0^2+R^2)} - \gamma\left[\frac{X_0 R}{X_0^2+R^2} - \tan^{-1}\frac{R}{X_0}\right]\right\} \tag{11}$$

Again the arctangent term must be limited to values in the first quadrant to obtain the physically significant solution for positive values of X_0 in Figure 12.

19. Having solved this simpler problem, it is now possible to return to the more complex problem in Figure 11 with which we started, and to make a proper selection in the case of multiple solutions to obtain the physically significant value. It is, of course, impossible here to give the details of the integrations involved in evaluating equation 6. About all that we can do is to mention a few of the high points and then give the result. To integrate equation 6 over ϕ for the shaded semicircular area in Figure 11, it will be found, after a change in variables, that this integral can be split up into three integrals having the following forms.

$$dF = B\,ds'X_0\left[A\int_0^R\int_{\Delta_2-\frac{\pi}{2}}^{\Delta_2+\frac{\pi}{2}}\frac{d\nu\,dh}{(E-\gamma_2\sin\nu)^2} - \gamma_1\cos(\Delta_1-\Delta_2)\int_0^R\int_{\Delta_2-\frac{\pi}{2}}^{\Delta_2+\frac{\pi}{2}}\frac{\sin\nu\,d\nu\,dh}{(E-\gamma_2\sin\nu)^2}\right. \tag{12}$$

$$\left. -\gamma_1\sin(\Delta_1-\Delta_2)\int_0^R\int_{\Delta_2-\frac{\pi}{2}}^{\Delta_2+\frac{\pi}{2}}\frac{\cos\nu\,d\nu\,dh}{(E-\gamma_2\sin\nu)^2}\right]$$

The various constants in this equation have the following values:

$$
\begin{aligned}
A &= h(\alpha X_0 + \beta Y_0 + \gamma Z_0) & E &= X_0^2 + Y_0^2 + Z_0^2 + h^2 \\
C &= \beta h^2 & F &= 2 Y_0 h \\
D &= \gamma h^2 & G &= 2 Z_0 h \\
\tan \Delta_1 &= \frac{D}{C} & \tan \Delta_2 &= \frac{G}{F} \\
Y_1 &= \sqrt{C^2 + D^2} & Y_2 &= \sqrt{F^2 + G^2}
\end{aligned}
\tag{13}
$$

All these integrals are of well known forms, and hence, can be directly integrated. After the indicated integrations are carried out, the limits inserted, and various simplifications made, the following result was found:

$$
dF = 2 B ds' X_0 \left\{ \int_0^R \frac{AEG - CFG + DF^2 - DE^2}{(E^2 - Y_2^2)(E^2 - F^2)} \, dh + \int_0^R \frac{AE - CF - DG}{(E^2 - F^2)^{3/2}} \tan^{-1} \frac{\sqrt{E^2 - Y_2^2}}{-2 Z_0 h} \, dh \right\}.
\tag{14}
$$

We also have an arctangent term appearing in this equation. The previously mentioned difficulty in solving this problem lies in the proper choice of values for this term. To determine which is the physically significant value, we set $Y_0 = Z_0 = 0$ and then compare the resulting equation with equation 8 for the simpler problem. It soon becomes clear that the arctangent term must be $\pi/2$, so that for arbitrary positive values of Z_0 suggested by Figure 11, this arctangent term must be restricted to values in the second quadrant.

20. The remaining task is that of integrating this equation over h between the limits of zero and R. To do this, the various terms in this equation, which are functions of h, must be written out in detail and, where possible, expressed in terms of other simplifying constants, which are not functions of h. When this is done, the resulting equation for dF can be expressed in terms of the following five integrals:

$$
\begin{aligned}
dF = 2B\, ds' X_0 \Bigg\{ & 2\alpha X_0 Z_0 \int_0^R \frac{(a+h^2)h^2\,dh}{\left[(a+h^2)^2 - bh^2\right]\left[(a+h^2)^2 - ch^2\right]} \\
& + 2\beta Y_0 Z_0 \int_0^R \frac{(a-h^2)h^2\,dh}{\left[(a+h^2)^2 - bh^2\right]\left[(a+h^2)^2 - ch^2\right]} \\
& - \gamma \int_0^R \frac{(d+ch^2+h^4)h^2\,dh}{\left[(a+h^2)^2 - bh^2\right]\left[(a+h^2)^2 - ch^2\right]} \qquad (15) \\
& + \alpha X_0 \int_0^R \frac{(a+h^2)h}{\left[(a+h^2)^2 - bh^2\right]^{3/2}} \tan^{-1} \frac{\sqrt{(a+h^2)^2 - bh^2}}{-\sqrt{b-c}\, h}\, dh \\
& + (\beta Y_0 + \gamma Z_0) \int_0^R \frac{(a-h^2)h}{\left[(a+h^2)^2 - bh^2\right]^{3/2}} \tan^{-1} \frac{\sqrt{(a+h^2)^2 - bh^2}}{-\sqrt{b-c}\, h}\, dh \Bigg\}.
\end{aligned}
$$

In this equation the terms not previously defined have the values

$$a = X_0^2 + Y_0^2 + Z_0^2 \qquad d = X_0^4 + Y_0^4 - Z_0^4 + 2X_0^2Y_0^2$$

$$b = 4(Y_0^2 + Z_0^2) \qquad e = 2(X_0^2 - Y_0^2) \tag{16}$$

$$c = 4Y_0^2$$

To evaluate these integrals, the first three must be separated into simpler integrals, and the last two integrated by parts. Each of the resulting integrations was always checked by differentiation before proceeding to be sure that no mistakes appeared. These integrations are all too long to be repeated here. It will have to be sufficient to say that after these integrations were completed, and the results substituted in equation 15, seventeen terms resulted. Fortunately two of these reduce to zero, six cancel out, and other simplifications are possible. The resulting equation for dF in one of its simpler forms is then

$$dF = B\,ds'X_0\left\{\frac{\pi\alpha}{4X_0} - \frac{\pi(\beta Y_0 + \gamma Z_0)}{4(Y_0^2+Z_0^2)} + \frac{(\alpha Z_0 - \gamma X_0)}{2X_0\sqrt{X_0^2+Z_0^2}}\tan^{-1}\frac{2R\sqrt{X_0^2+Z_0^2}}{a-R^2}\right.$$

$$+ \frac{\gamma Y_0 - \beta Z_0}{4(Y_0^2+Z_0^2)}\log_e\frac{a+R^2+2Y_0R}{a+R^2-2Y_0R} \tag{17}$$

$$\left. - \frac{\alpha(Y_0^2+Z_0^2)(a-R^2) - X_0(\beta Y_0+\gamma Z_0)(a+R^2)}{2X_0(Y_0^2+Z_0^2)\sqrt{(a+R^2)^2-bR^2}}\tan^{-1}\frac{\sqrt{(a+R^2)^2-bR^2}}{-2Z_0R}\right\},$$

If attempts are made to reduce this equation to equation 9 for the simpler case in which $Y_o = Z_o = 0$, terms of indeterminate forms will be found to appear. When the derivatives are investigated, however, these terms will be found to have values which reduce the previous equation to equation 9. The a and the b have the values already indicated in equation 16. The first arctangent term may lie in either the first or second quadrants depending upon whether a is larger or smaller than R^2, but the second arctangent term is restricted to values in the second quadrant for positive values of Z_o suggested by Figure 11.

21. A similar equation can also be found for the unshaded semicircular area in Figure 11. The procedure is the same as that for the shaded semicircular area, and an equation of the same form as equation 6 is obtained, but the limits on the angle ϕ now extend from $\pi/2$ to $3/2\,\pi$. Integration of this equation over the angle ϕ and substitution of the limits again gives equation 14, but each of the two integrals now has a negative sign in front. Clearly if we proceed to integrate this equation, we would find a result which is just the negative of the value found for the shaded semicircular area. Obviously such a solution has no physical significance. The key to this difficulty again lies in the multiple values provided by the arctangent term. If we compare this equation with the simpler problem for which $Y_o = Z_o = 0$, we find that the arctangent term must be restricted to values in the first quadrant for positive values of Z_o suggested by Figure 11. After the integration over ϕ is carried out and the limits inserted, we find that dF must then be

$$dF = 2Bds'X_o \left\{ -\int_0^R \frac{AEG - CFG + DF^2 - DE^2}{(E^2 - Y_2^2)(E^2 - F^2)} \right.$$

$$\left. + \int_0^R \frac{AE - CF - DG}{(E^2 - F^2)^{3/2}} \tan^{-1} \frac{\sqrt{E^2 - Y_2^2}}{2Z_o h}\, dh \right\}, \tag{18}$$

where now the arctangent term is restricted to values in the first quadrant for positive values of Z_o suggested by Figure 11.

22. The integration over h can be done in much the same way as before, the significant differences being that changes in signs appear in some of the terms. An equation similar to equation 15 will be found, in which the first three integrals are identical but opposite in sign, and the last two identical except for a positive sign in the denominator of the arctangent term. The only new integrals appearing for this semicircular area are then the last two. Substitution of the values for all these integrals in equation 15 after simplification yields the following equation.

$$dF = B\,ds'X_0\left\{\frac{\pi\alpha}{4X_0} - \frac{\pi(\beta Y_0+\gamma Z_0)}{4(Y_0^2+Z_0^2)} - \frac{\alpha Z_0-\gamma X_0}{2X_0\sqrt{X_0^2+Z_0^2}}\tan^{-1}\frac{2R\sqrt{X_0^2+Z_0^2}}{a-R^2}\right.$$

$$-\frac{\gamma Y_0-\beta Z_0}{4(Y_0^2+Z_0^2)}\log_e\frac{a+R^2+2Y_0R}{a+R^2-2Y_0R} \qquad (19)$$

$$\left.-\frac{\alpha(Y_0^2+Z_0^2)(a-R^2)-X_0(\beta Y_0+\gamma Z_0)(a+R^2)}{2X_0(Y_0^2+Z_0^2)\sqrt{(a+R^2)^2-bR^2}}\tan^{-1}\frac{\sqrt{(a+R^2)^2-bR^2}}{2Z_0R}\right\}.$$

Again the first arctangent term may lie in either the first or second quadrants depending upon whether a is larger or smaller than R^2, but the second arctangent term is now restricted to the first quadrant for positive values of Z_0 suggested by Figure 11.

23. If equations 17 and 19 are compared now, it will be noted that the third and fourth terms of each equation differ in sign and the denominators of the second arctangent term differ in sign. These equations represent the first of the very general contributions of this report, for, as far as we could find, these equations are new to the photometry of semicircular radiating surfaces. It is now possible, by utilizing these equations, to calculate the total flux falling on an elementary receiving area from a semicircular radiating surface, when the elementary receiving surface has arbitrary coordinates and a surface normal with arbitrary direction cosines. These equations are of a very

general form, so that all similar equations, for which the semicircular source is confined to one of the coordinate planes at the center of the coordinate system become special cases of these equations in which either the coordinates or the direction cosines of the surface normal of the elementary receiving area have particular values.

24. As independent checks on the correctness of equations 17 and 19, two problems indicated by Figures 13 and 14 were solved. In each case, the contributions of the shaded and unshaded semicircular areas to the total flux falling on the elementary receiving area was determined. Both of these problems will be recognized clearly as special cases of the more general problem already solved, for which either Y_0 or Z_0 has been made equal to zero. When either Y_0 or Z_0 is set equal to zero in the more general equations already presented, the resulting equations were in complete agreement with the equations obtained by solving the problems suggested by these two figures. Symmetry tests were also applied and, in each case, equations 17 and 19 had the desired forms.

IV. GENERAL EQUATION FOR RADIATION FIELD OF CIRCULAR SURFACE SOURCE

A. Final Form

25. The second problem, which has concerned us in this report, was that of calculating the total flux falling on an elementary receiving area from a circular source when the elementary receiving area had arbitrary coordinates and a surface normal with arbitrary direction cosines. Since we have already determined the contributions of each of the semicircular areas in Figure 11, the contribution of the complete circular area will then simply be the sum of the contributions of the individual semicircular areas. This follows because of the assumed small wave length and random phases and small particle size and noninteraction of particles which were necessary before surface integrations could be done. Returning to equations 17 and 19, we can then write the following equation for the total flux falling on the elementary receiving area from a circular source when the elementary receiving area has arbitrary coordinates and a surface normal with arbitrary direction cosines,

$$dF = \frac{\pi B ds' X_0}{2}\left\{\frac{\alpha}{X_0}\left[1-\frac{a-R^2}{\sqrt{(a+R^2)^2-bR^2}}\right]+\frac{\beta Y_0+\gamma Z_0}{Y_0^2+Z_0^2}\left[\frac{a+R^2}{\sqrt{(a+R^2)^2-bR^2}}-1\right]\right\}, \quad (20)$$

the α and the b again having the values indicated in equation 16. This equation represents the second of the more general contributions of this paper. Again, this equation is also very general in form so that all similar equations, for which the circular source is confined to one of the coordinate planes at the center of the coordinates, become special cases of this equation, in which either the coordinates or the direction cosines of the surface normal of the elementary receiving area have particular values. This equation, although not specifically written in the literature, can, after a redefinition of terms, be inferred from Fock's components, (reference (g)), of his "light vector", if one writes down the light vector for an arbitrary position and orientation.

B. Secondary Check

26. As an independent check on the correctness of the equation for the circular source, the problem was solved in a completely different manner suggested by Figure 15. In this case, the elementary receiving area is still permitted to have arbitrary coordinates and a surface normal with arbitrary direction cosines, but it is restricted to an arbitrarily oriented plane normal to the circular surface and through its center, which divides the circular area into two semicircular areas as indicated. As in the previous problem, one then calculates the total flux falling on the elementary receiving areas from both the shaded and unshaded semicircular areas and simply adds them. The integrals obtained in this case are much simpler than those found in solving the problem in the previous way. In fact, the problem of determining the total flux falling on the elementary receiving area from the complete circular source was first solved in this manner because the integrations were much simpler. The integral equation, which must be solved for the shaded semicircular area in Figure 15, will be of the same form as that indicated in equation 6, but the upper and lower limits for the variable α must be replaced by the values $\tan^{-1} Y_0/Z_0$ and $\tan^{-1} Y_0/Z_0 - \pi$. After a change in variables, this integral can be separated into the same three integrals of standard forms found in solving the problem in the other way, which can be evaluated, and the new limits on α substituted. When the new limits are substituted, the forms of the equations for the two problems then diverge. There is again an ambiguity in sign which appears and must be clarified by comparing the integrals of the general problem with those for the simpler problem for which Y_0 and Z_0 are set equal to zero. Finally one arrives at four integrals of nonstandard forms, which are simpler than those corresponding to the previous solution of the problem, (see equation 15), and which must be integrated over the variable h. After this integration is carried out, the limits substituted and certain substitutions made, the resulting equation for the total flux contributed by the shaded semicircular area in Figure 15 was found to be

$$dF = \frac{B\,ds'X_0}{4}\left\{\frac{\pi\alpha}{X_0}\left[1-\frac{a-R^2}{\sqrt{(a+R^2)^2-bR^2}}\right]+\frac{\pi(\beta Y_0+\gamma Z_0)}{Y_0^2+Z_0^2}\left[\frac{a+R^2}{\sqrt{(a+R^2)^2-bR^2}}-1\right]\right.$$

$$\left.-\frac{\beta Z_0-\gamma Y_0}{Y_0^2+Z_0^2}\log_e\frac{a+R^2+2R\sqrt{Y_0^2+Z_0^2}}{a+R^2-2R\sqrt{Y_0^2+Z_0^2}}+\frac{2(\beta Z_0-\gamma Y_0)}{X_0\sqrt{Y_0^2+Z_0^2}}\tan^{-1}\frac{2X_0R}{a-R^2}\right\}, \tag{21}$$

the a and the b again having the values given by equation 16. This equation also appears to be new to the photometry of semi-circular areas. The total flux contributed by the unshaded area in Figure 15 can be found in the same way and will be found to be equal to the same equation with the signs of the last two terms reversed. The total contribution of the circular area is then simply twice that of the first two terms of the last equation, which is exactly what we found by solving the problem in the previous way. Symmetry tests were also applied to this equation for the circular source, and in each case the equation gave the desired results.

V. SPECIAL FORMS OF THE EQUATION FOR THE SEMICIRCULAR SURFACE SOURCE

27. Another method for checking the correctness of these equations, as well as those of others, is to insert particular values for the coordinates and the direction cosines of the surface normal of the elementary receiving area, and then see if the resulting equation will reduce to the particular forms published by others. Since only three equations were found for the semicircular source, we will consider the semicircular source first.

A. Walsh's Formula

28. If we consider these particular formulae in the order of decreasing complexity, the most complex problem would be that given in Figure 10 which was considered by Walsh. From this figure, it is clear that if the general equation 17 is to reduce to this particular case, we must make the following substitutions:

$$Y_0 = 0,\ \alpha = \beta = 0,\ Z_0 = -Z_0,\ \gamma = -1.$$

Inserting these values in equation 17, we find that dF is given by

$$dF = B\,ds'\frac{X_o}{Z_o}\left\{\frac{X_o^2+Z_o^2+R^2}{2\sqrt{(X_o^2+Z_o^2+R^2)^2-4Z_o^2R^2}}\tan^{-1}\frac{\sqrt{(X_o^2+Z_o^2+R^2)^2-4Z_o^2R^2}}{2Z_oR}\right. \tag{22}$$

$$\left.+\frac{Z_o}{2\sqrt{X_o^2+Z_o^2}}\tan^{-1}\frac{2R\sqrt{X_o^2+Z_o^2}}{X_o^2+Z_o^2-R^2}-\frac{\pi}{4}\right\}.$$

If we make use of the relation that

$$2\tan^{-1}\frac{M}{P} = \tan^{-1}\frac{2PM}{P^2-M^2}, \tag{23}$$

the previous equation can be written in the final form in which Walsh, (reference (j)), has written it. Walsh's equation is then

$$dF = B\,ds'\frac{X_o}{Z_o}\left\{\frac{X_o^2+Z_o^2+R^2}{\sqrt{(X_o^2+Z_o^2+R^2)^2-4Z_o^2R^2}}\tan^{-1}\sqrt{\frac{X_o^2+Z_o^2+R^2-2Z_oR}{X_o^2+Z_o^2+R^2+2Z_oR}}\right. \tag{24}$$

$$\left.+\frac{Z_o}{\sqrt{X_o^2+Z_o^2}}\tan^{-1}\frac{R}{\sqrt{X_o^2+Z_o^2}}-\frac{\pi}{4}\right\}.$$

This equation is of course correct.

B. Bordoni's Formulae

29. The next simplest formula is that of Bordoni for the particular case described by Figure 9. This problem is a special case of the previously presented Walsh formula for which Z_o also is equal to zero. If we set Z_o equal to zero in equation 22, the equation reduces to the following indeterminate form:

$$dF = Bds'X_o\left\{\alpha + \frac{1}{X_o}\tan^{-1}\frac{R}{X_o} - \alpha\right\}.$$

To reduce the two indeterminate terms in this equation to a single indeterminate term of the form 0/0, the first and last terms can be written in the form of a single term. To determine the value of this term at $Z_o = 0$, it becomes necessary to investigate the derivates of the numerator and denominator in the conventional manner. When this is done, we find that the final expression for dF, when Z_o approaches zero, is given by

$$dF = Bds'\left\{\tan^{-1}\frac{R}{X_o} - \frac{RX_o}{X_o^2+R^2}\right\}. \qquad (25)$$

If we leave out the ds', this equation, except for the manner in which Bordoni defines the quantity B, is equivalent to Bordoni's equation, (reference (c)), for the flux density, and it is of course correct.

30. The last formula found for the semicircular radiating surface is Bordoni's equation for the case illustrated by Figure 8. This is a very simple case, and if our general equation is to reduce to this particular case, the following substitutions must be made:

$$\alpha = 1, \quad \beta = \gamma = 0, \quad Y_o = Z_o = 0.$$

These substitutions can be made directly in equation 17 without any difficulties. When this is done, dF will be found to have the value

$$dF = \frac{\pi B ds'}{2}\ \frac{R^2}{X_o^2+R^2}. \qquad (26)$$

Again if we leave out the ds', this equation (except for the manner in which B is defined) is directly reducible to Bordoni's equation, (reference (c)), for the flux density. This equation is also correct.

VI. SPECIAL FORMS OF THE EQUATION FOR THE CIRCULAR SURFACE SOURCE

A. Bethe's Formula

31. In discussing special forms of the general equation for the circular source, we shall also discuss them in the order of decreasing complexity. The most complicated equation found in the literature for the circular surface source was that of Bethe described by Figure 7. Although our general formula has not been derived to cover this specific case, it can nevertheless be used if a simple translation of the coordinates along the Y_o axis is carried out. The Y in this figure, indicating the position of the center of the circular source, becomes the Y_o for the translated coordinates, indicating the Y coordinate of the elementary receiving area. If then, we set Z_o equal to zero in equation 20, we should obtain an equation equivalent to Bethe's equation. Doing this we find that

$$dF = \frac{\pi B ds'}{2}\alpha\left[1 - \frac{X_o^2+Y_o^2-R^2}{\sqrt{(X_o^2+Y_o^2+R^2)^2-4Y_o^2R^2}}\right]$$

$$+\frac{\pi B ds'}{2}\frac{X_o}{Y_o}\beta\left[\frac{X_o^2+Y_o^2+R^2}{\sqrt{(X_o^2+Y_o^2+R^2)^2-4Y_o^2R^2}}-1\right]. \tag{27}$$

If the direction cosines in this equation are expressed in terms of Bethe's angles, and the ds' is left out, this equation will be found to be equivalent to Bethe's equation, (reference (i)), for the flux density.

B. Fock's Formulae

32. Fock's components of the "light vector", suggested by Figure 5, are the next simplest forms of our general equation for the circular source. To obtain equations for the total flux

falling on the elementary receiving area when it has arbitrary coordinates and is oriented in the three directions of the coordinate axes, we must make α, β, and γ in equation 20 equal to unity. Doing this,

$$
\begin{aligned}
dF_x &= \frac{\pi B ds'}{2}\left[1-\frac{a-R^2}{\sqrt{(a+R^2)^2-bR^2}}\right], \quad (\alpha=1)\\
dF_y &= \frac{\pi B ds'}{2}\,\frac{X_0 Y_0}{Y_0^2+Z_0^2}\left[\frac{a+R^2}{\sqrt{(a+R^2)^2-bR^2}}-1\right], \quad (\beta=1)\\
dF_z &= \frac{\pi B ds'}{2}\,\frac{X_0 Z_0}{Y_0^2+Z_0^2}\left[\frac{a+R^2}{\sqrt{(a+R^2)^2-bR^2}}-1\right]. \quad (\gamma=1)
\end{aligned}
\tag{28}
$$

These equations, except for the constant factor π B ds', are equivalent to Fock's equations, (reference (g)), for his three components of the "light vector". Conversely, if we form a scalar product of the unit vector along the surface normal of the elementary receiving area with the components of Fock's "light vector", we obtain equation 20 except for the constant factor π B ds'. Fock had suggested that such an equation might be written, but he never actually wrote it down because of his interest in the three simpler cases. It would seem wiser to write these equations in the forms suggested by equations 20 and 28, for these equations are much closer to quantities which would be encountered in actual experimental measurements.

C. Walsh's Formula

33. The next simplest problem is Walsh's problem suggested by Figure 6. This problem is a special case of the equation for dF_y in equation 28 for which Z_0 is zero. If we make $Z_0 = 0$ in this equation, we find that

$$dF_y = \frac{\pi B ds'}{2}\,\frac{X_0}{Y_0}\left[\frac{X_0^2+Y_0^2+R^2}{\sqrt{(X_0^2+Y_0^2+R^2)^2-4Y_0^2R^2}}-1\right]. \tag{29}$$

This equation should be Walsh's equation, but a discrepancy in sign will be found between the last term in the brackets and the corresponding term in Walsh's equation, (reference (h)). This has been acknowledged as an error by Walsh and has been corrected in the new edition of his book.

D. Hillebrand's Formula

34. Another of the simpler forms of this general equation is that suggested by Figure 3 given by Hillebrand, (reference (f)), when the elementary receiving area is restricted to the X_0 axis and is permitted to have a surface normal with arbitrary direction cosines. To arrive at Hillebrand's equation, it is necessary to set $Y_0 = Z_0 = 0$ in equation 20. Doing this, we find that the second term is of the indeterminate form 0/0. When, however, the derivates are investigated, the following equation will be found:

$$dF = \pi B ds' \alpha \frac{R^2}{X_0^2 + R^2} \tag{30}$$

This equation will be found to be identical with that given by Hillebrand, (reference (f)).

E. Lambert's Formulae

35. As suggested earlier, one of the most often repeated equations for the circular source is that given for Figure 2. This equation was first giver by Lambert. To reduce our more general equation to Lambert's equation, it is necessary to set $\alpha = 1$, and $Z_0 = 0$. Making these substitutions in equation 20, we find that

$$dF_X = \frac{\pi B ds'}{2} \left[1 - \frac{X_0^2 + Y_0^2 - R^2}{\sqrt{(X_0^2 + Y_0^2 + R^2)^2 - 4Y_0^2 R^2}} \right]. \tag{31}$$

Except for the constant term B which was unity in Lambert's equation and the ds', this equation is equivalent to Lambert's equation for the flux density, (reference (b)), on the elementary receiving area, when it is oriented with its surface normal perpendicular to the source and is displaced from the axis as suggested by Figure 2.

36. A special case of Lambert's equation is that given by Foote, suggested by Figure 4. Obviously to reduce Lambert's equation to Foote's equation, it is necessary to set $Y_o = R$. Doing this, we find that

$$dF_x = \frac{\pi B ds'}{2}\left[1 - \frac{X_o}{\sqrt{X_o^2 + 4R^2}}\right] \qquad (32)$$

If the ds' is omitted, this equation is the same as Foote's original equation for the flux density and it is of course correct.

37. The last and simplest form of the more general equation is that given first by Lambert and suggested by Figure 1. To reduce this equation to a form comparable with Lambert's equation, we must as suggested by Figure 1 set $\alpha = 1$ and $Y_o = Z_o = 0$. Doing this,

$$dF_x = \pi B ds' \frac{R^2}{X_o^2 + R^2} \quad . \qquad (33)$$

This is Lambert's equation, (reference (b)), for the flux density if the B is set equal to unity and the ds' is omitted. Except for the factor two, it is also equivalent to the corresponding equation 26 for the semicircular source.

VII. EXAMPLES OF RADIATION FIELDS OF SEMICIRCULAR AND CIRCULAR SURFACE SOURCES

38. These equations offer the possibility of many types of field calculations for both types of sources. The examples which we are including in this report are of a very limited nature and hence do not illustrate the full versatility of these equations. They may, however, be of some general interest and illustrate the type of data which can be obtained from these equations.

A. Semicircular Source

39. For the semicircular source, we have considered the problem of evaluating the total flux on the elementary receiving area

when it is confined to a plane perpendicular to the X_0 axis in Figure 11 and has its surface normal perpendicular to this plane. To obtain the equation for the total flux dF for this particular problem, it is necessary to make $\alpha = 1$ (or $\beta = \gamma = 0$) in equation 17. Doing this, we find that

$$dF = B\,ds' \left\{ \frac{\pi}{4} + \frac{Z_0}{2\sqrt{X_0^2 + Z_0^2}} \tan^{-1} \frac{2R\sqrt{X_0^2+Z_0^2}}{a - R^2} - \frac{(a-R^2)}{2\sqrt{(a+R^2)^2 - bR^2}} \tan^{-1} \frac{\sqrt{(a+R^2)^2 - bR^2}}{-2Z_0 R} \right\} \quad (34)$$

The particular plane chosen was at a distance from the semi-circular source equal to its radius R. If we now introduce plane polar coordinates in the (Y_0, Z_0) plane containing the elementary receiving area and express both X_0 and r in terms of the radius R of the semicircular source, then for purposes of calculation, the previous equation can be written in the following form.

$$\frac{dF}{B\,ds'} = \frac{\pi}{4} + \frac{\bar{r}\cos\alpha}{2\sqrt{1+\bar{r}^2\cos^2\alpha}} \tan^{-1} \frac{2\sqrt{1+\bar{r}^2\cos^2\alpha}}{\bar{r}^2} - \frac{\bar{r}^2}{2\sqrt{\bar{r}^4+4}} \tan^{-1} \frac{\sqrt{\bar{r}^4+4}}{-2\bar{r}\cos\alpha}. \quad (35)$$

In arriving at numerical values from these formulae, intervals in α of 10° were chosen and along each of these directions values of $\bar{r}$ from .5 to 10 were inserted. Along each of these directions, graphs were then plotted and values of $\bar{r}$ found for which dF/B ds' had chosen constant values. A polar diagram of the form indicated in Figure 16 was then plotted from this data. In the region surrounding the semicircular disk drawn to scale at the center are plotted these isophotopic lines along which the quantity dF/B ds' is constant, and the value along each of these lines is indicated. These isophotopic lines resemble distorted circles which are symmetric about the Z_0 axis.

B. Circular Source

40. For the circular source, we considered the same problem of evaluating the total flux on the elementary receiving area when it was confined to a plane perpendicular to the X_0 axis in Figure 11 and had its surface normal perpendicular to this plane. For this problem, α, in equation 20 becomes unity so that dF is given by

$$dF = \frac{\pi B ds'}{2}\left[1 - \frac{a - R^2}{\sqrt{(a+R^2)^2 - bR^2}}\right]. \tag{36}$$

As suggested earlier (see equation 31), this is Lambert's equation which appeared in 1760. For purposes of calculation, polar coordinates were again introduced in the (Y_o, Z_o) plane, and the particular plane chosen was again at a distance from the source equal to its radius R. If the distance r is again expressed in units of R, then the previous equation can for purposes of calculation be written as follows:

$$dF/B ds' = \frac{\pi}{2}\left[1 - \frac{\bar{r}^2}{\sqrt{\bar{r}^4+4}}\right] \tag{37}$$

Obviously, if we plot the quantity dF/B ds', we must then obtain circles in the (Y_o, Z_o) plane as shown in Figure 17. In this figure the circular disk is again drawn to scale at the center of the figure, and the isophotopic lines have their corresponding values of the quantity dF/B ds' marked directly on them.

41. Before leaving this figure, it might be of interest to note that the usual $\cos^4$ law quoted in illumination calculations at off axis positions in the focal plane of a perfect lens system turns out to be an approximation of Lambert's exact equation which is obtained by assuming that $R^2 << X_o^2 + r^2$. If this assumption is introduced into equation 36 and the radical expanded up to second order terms, we find

$$dF = \pi B ds' \frac{X_o^2 R^2}{(X_o^2 + r^2)^2} = \frac{B ds'}{X_o^2}(\pi R^2)\cos^4\omega, \tag{38}$$

which is the $\cos^4$ law as it is usually written.

42. Isophotopic plots of the form suggested by Figures 16 and 17 can be quite useful in making rapid evaluations of the total flux dF or the flux density dF/ds'. All one needs to know to utilize

such a graph are the magnitude of B and the coordinates of the point in question and then the flux density dF/ds' is uniquely determined. If in addition the area ds' is known, then also dF can be found.

43. Calculations of this type are quite readily extended to many types of complex surfaces, for all one needs to know are the values of the coordinates (X_0, Y_0, Z_0) and the direction cosines (α, β, γ) of the surface normal over this surface and then both dF and dF/ds' can be evaluated. The only precaution in the use of these equations is that of being certain at each point on the surface that the plane containing the elementary receiving area does not intersect the source, for if it does, these equations, except in very limited cases, are no longer applicable.

VIII. CONCLUSIONS AND ACKNOWLEDGEMENTS

44. In this report we have extended the treatment given to the rectangular source in a previous report to cover the semicircular and circular sources. Assuming that both the semicircular and circular sources radiate uniformly over their surfaces and obey Lambert's Cosine Law, we have presented equations giving the total flux falling on an elementary receiving area, when the source is located at the origin in one of the coordinate planes, and the elementary receiving area has arbitrary coordinates and a surface normal with arbitrary direction cosines. After a simple translation and rotation of axes, these same equations can also be used for the equally important problem in which the elementary receiving area is located at the origin of the coordinates in one of the coordinate planes, and the semicircular or circular sources has arbitrary coordinates and a surface normal with arbitrary direction cosines. With the equations in this form, it is then possible to calculate the total flux falling on the elementary receiving area in the presence of an array of semicircular and circular surface sources having arbitrary coordinates and orientation. In the previous report (NavOrd 2980), we have presented similar equations for the rectangular surface source. Having these three equations for these three different sources (rectangular, semicircular and circular), it is then conceivable that the radiation fields of more complex surface sources might be approximated by dividing them up into surface areas of these three types and adding together all the separate contributions to obtain the radiation field of the complex target. In carrying out these calculations, mechanized computing facilities will have to be utilized since the calculations even for these simpler sources become quite tedious by hand methods.

45. These equations are also directly applicable to problems for which the elementary receiving area becomes the source of flux, and the semicircular and circular areas the receiving areas. Thus one can, utilizing these equations, calculate the total flux entering a semicircular or circular aperture from a small elementary source area ds, when either the source area ds or the semicircular and circular areas have arbitrary coordinates and surface normals with arbitrary direction cosines. For example, if as in Figure 1, the source element ds is parallel to the plane of the circular receiving area and lies on its axis ($\alpha = 1$, $Y_o = Z_o = 0$), the total flux entering the circular aperture from equation 20 will be π B ds $\sin^2 U$, where U is half the angular opening as measured from the elementary source area ds. This is of course a well known result. Many similar equations can also be derived for special cases. Assuming again that the source area radiates uniformly over its surface and obeys Lambert's Cosine Law, the only limitation in the use of these equations is that of being certain that the plane containing the elementary source or receiving areas does not intersect the semicircular or circular areas. If it does, these equations, except under very special conditions, no longer are applicable, because the limits on the integrals are no longer true limits.

46. In this report, we have also given a history of the development of radiation formulae for the semicircular and circular sources. This history shows that nine equations could be found for the circular source, and only three for the semicircular source. There are no doubt others which we did not find. We have shown that each of these formulae can be derived from the general equations presented in this report by simply inserting particular values for the coordinates and the direction cosines of the elementary receiving area ds'. In one particular case, a translation of coordinates was necessary but the form of the equation was unaffected. As a result of the comparison of special forms of the general equations presented in this report with the equations of others describing simpler cases, we believe that all these formulae are correct except the formula of Jones, (reference (e)), for Figure 2 and the formula of Walsh, (reference (h)), for Figure 6. Jones has not had the opportunity of defending his formula, but Walsh has conceded the error in his formula.

47. Equations 17, 19, and 21 presented in this report for the semicircular source appear to be new. Even the simpler forms given by equations 9 and 11 were not found, but they are simple enough to derive so that others must have derived them earlier. The primary difficulty in deriving the more complex equations seems to have been the interpretation of the multiple roots, although the integrals themselves are complicated enough when one attempts

to evaluate them. Having derived the equations for the semi-circular source, the corresponding equation for the circular source is very readily obtained. This equation, although not specifically written by Fock, can nevertheless be arrived at from Fock's components of his "light vector", if the light vector is written for an arbitrary position and orientation, and certain definitions are brought into agreement.

48. This report, like the corresponding earlier report on the rectangular surface source, may be subject to criticism because it is completely theoretical. Objections can again be raised with regard to deviations which may result between calculated and measured values of dF due to the nonuniformity of the flux over the surface and to deviations from Lambert's Cosine Law. If these factors introduce marked deviations between measured and calculated values, it becomes necessary to initiate a series of detailed measurements to determine how B varies with position and angle of emergence and then try additional integrations. If these fail, about all that can be done then is to divide the larger area into smaller areas of rectangular, semicircular, and circular shapes which do approximately satisfy these conditions and determine the contributions of each and add.

49. In conclusion, it should perhaps be stated that these calculations have all been carried out using Lambert's method of surface integration. The primary reason for this was that we were interested in exploiting this method further to find out if these problems can be solved in this way. We find from these two reports that the method of surface integrations can be made to yield proper results for the rectangular, semicircular, and circular sources if sufficient care is taken in performing these detailed integrations and no mistakes are made. This method has the added advantage that all these calculations have been done with physically measurable quantities so that the final equations are also directly applicable to experimentally measured values.

50. The author is indebted to Walter Malmborg of the Applied Mathematics Branch of this Laboratory for the numerical calculations leading up to Figures 16 and 17. These values were obtained by using the mechanized computing facilities of this Laboratory. Mr. George Haskins of the Illustrations Group of this Laboratory was responsible for drawing Figures 16 and 17 and the author would like to acknowledge his interest in this work.

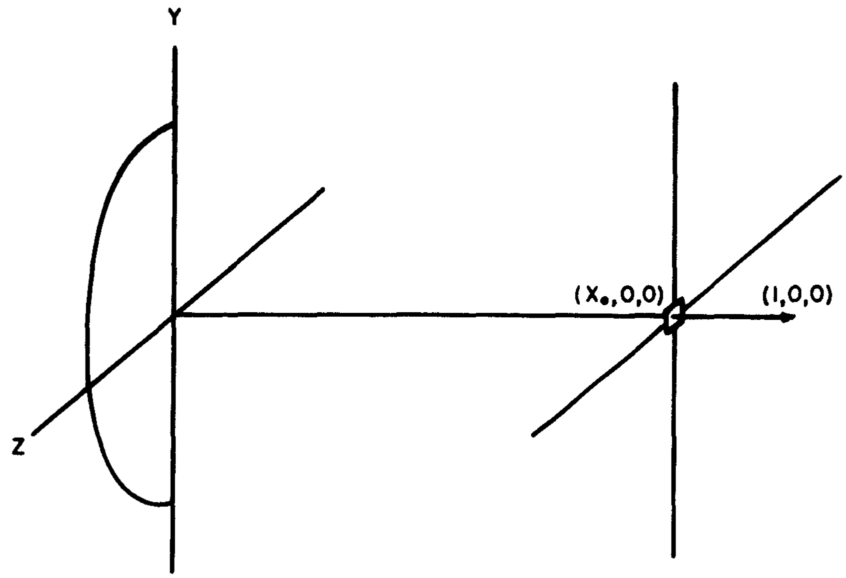

FIG. 1 LAMBERT'S FIRST PROBLEM

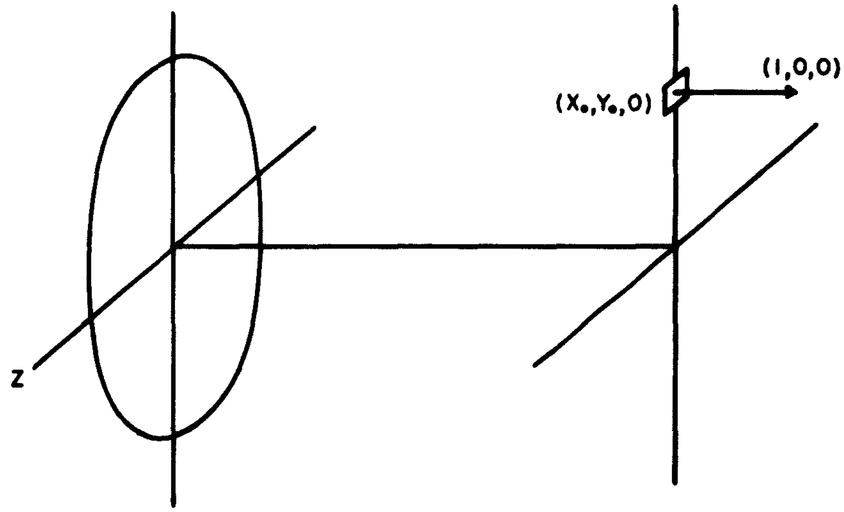

FIG. 2 LAMBERT'S SECOND PROBLEM

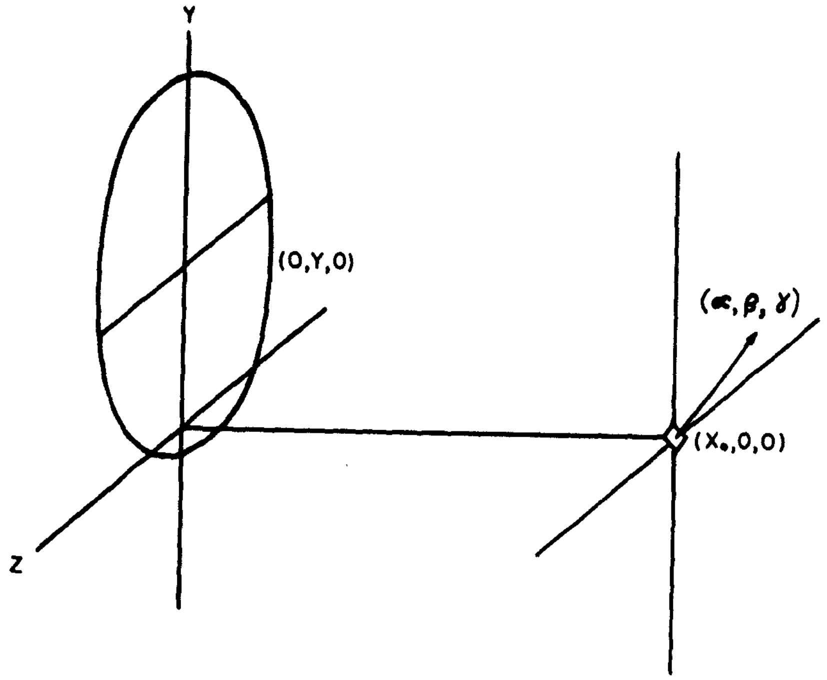

FIG. 3 HILLEBRAND'S PROBLEM

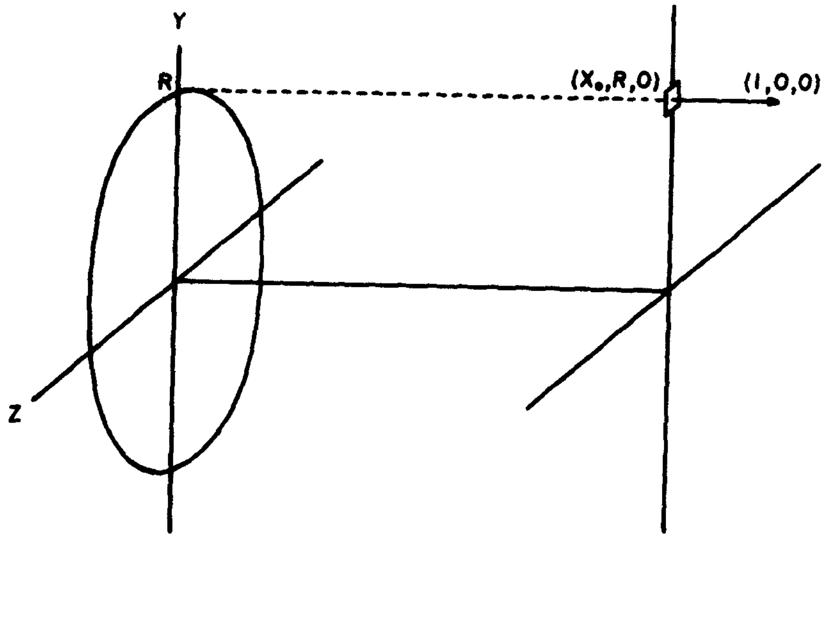

FIG. 4 FOOTE'S PROBLEM

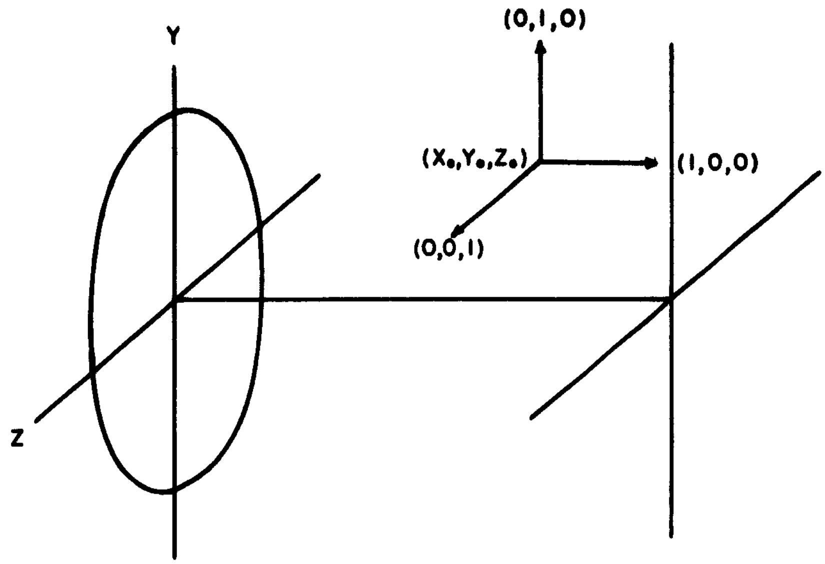

FIG. 5 FOCK'S PROBLEM

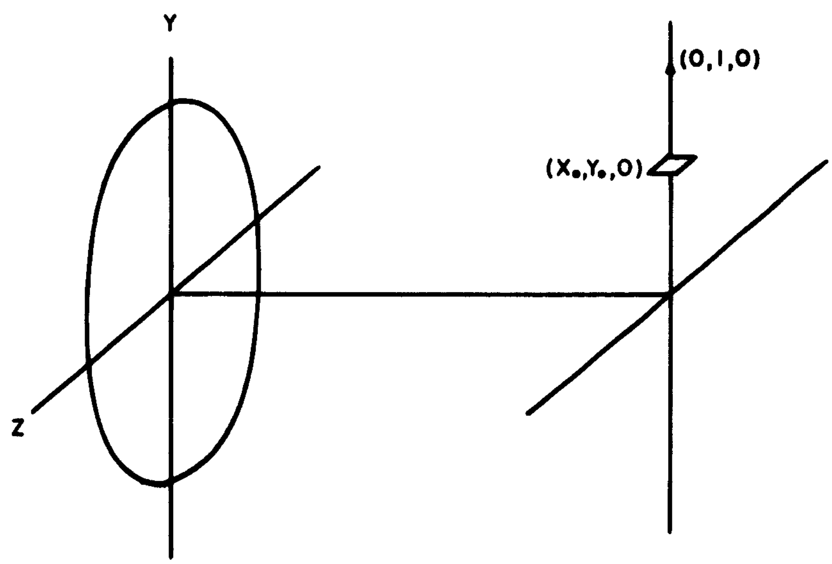

FIG. 6 WALSH'S PROBLEM

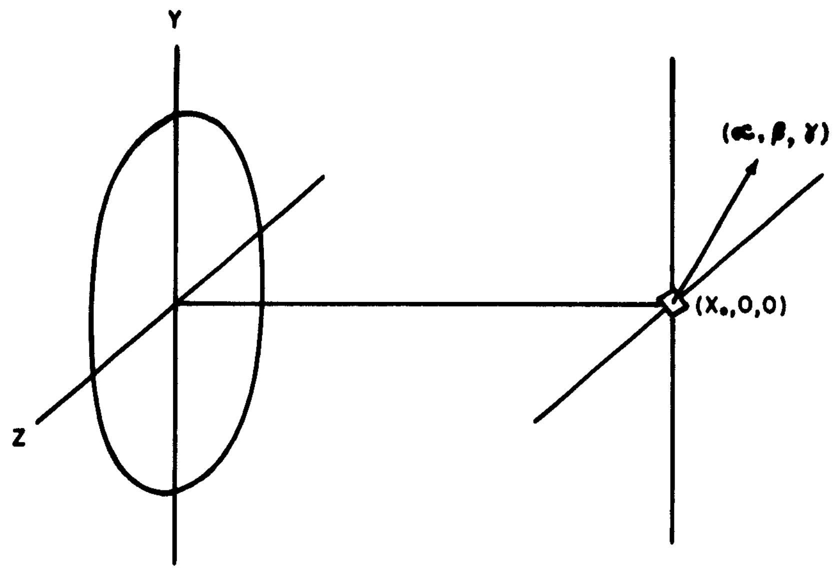

FIG. 7 BETHE'S PROBLEM

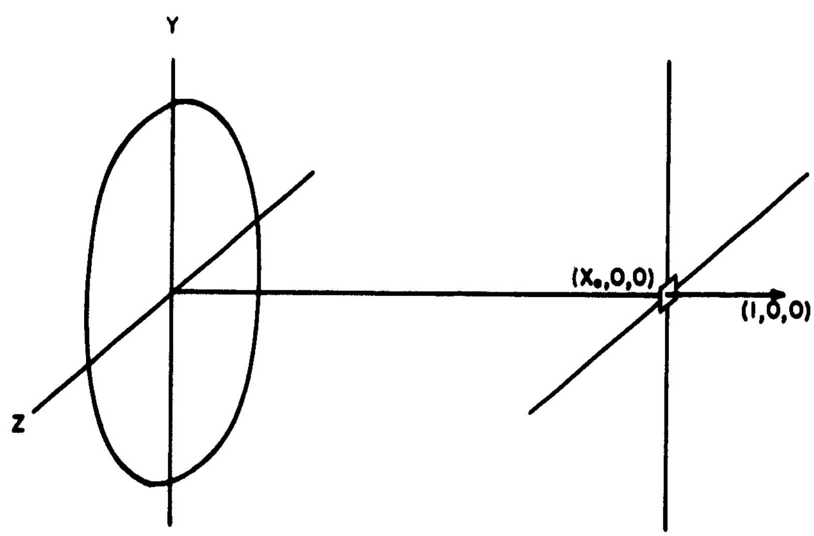

FIG. 8 BORDONI'S FIRST PROBLEM

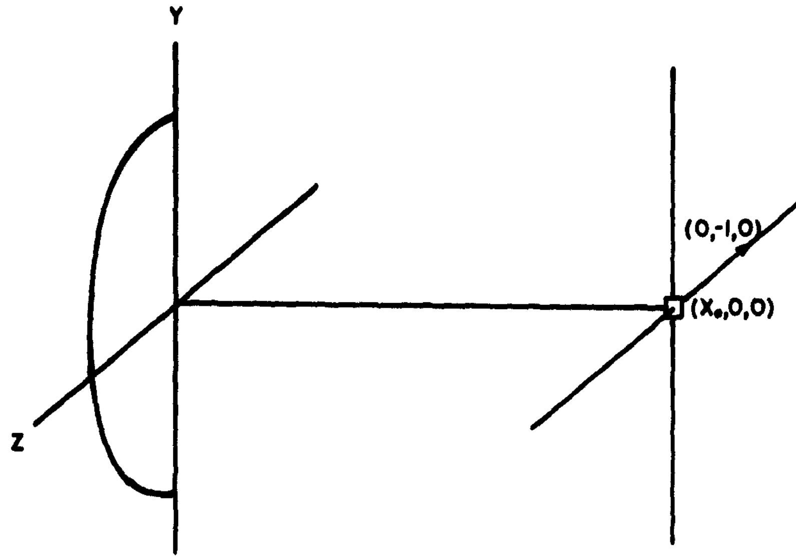

FIG. 9 BORDONI'S SECOND PROBLEM

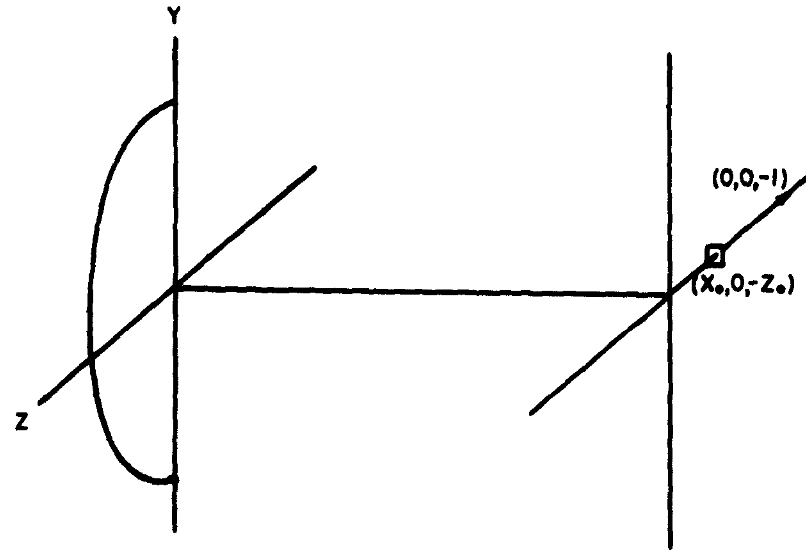

FIG. 10 WALSH'S PROBLEM

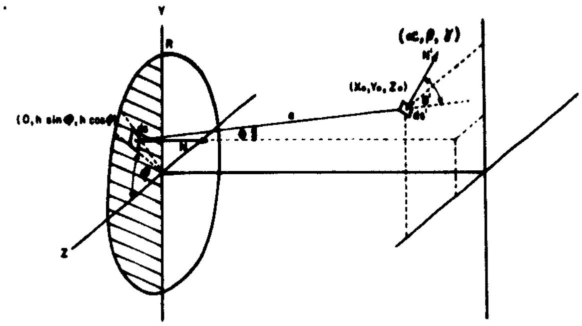

FIG. 11 FLUX FROM A SEMICIRCULAR SOURCE FALLING ON AN ELEMENTARY RECEIVING AREA WITH ARBITRARY COORDINATES HAVING A SURFACE NORMAL WITH ARBITRARY DIRECTION COSINES

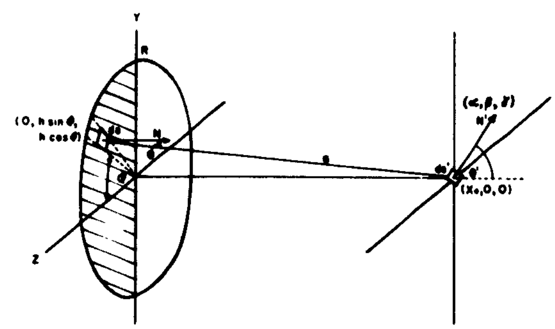

FIG. 12 FLUX FROM A SEMICIRCULAR SOURCE FALLING ON AN ELEMENTARY RECEIVING AREA HAVING ARBITRARY DIRECTION COSINES AND LIMITED TO THE X_0 AXIS

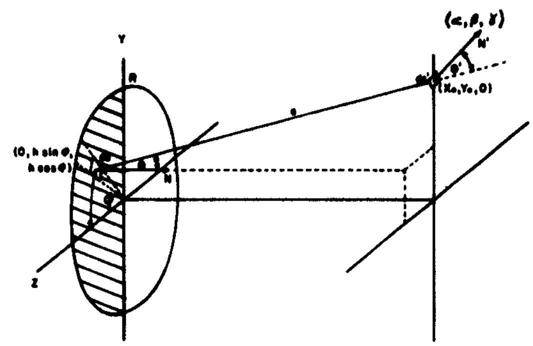

FIG. 13 FLUX FROM A SEMICIRCULAR SOURCE FALLING ON AN ELEMENTARY RECEIVING AREA HAVING ARBITRARY DIRECTION COSINES AND LIMITED TO THE Y_0 AXIS

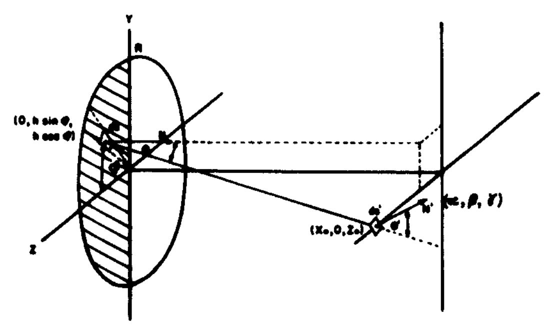

FIG. 14 FLUX FROM A SEMICIRCULAR SOURCE FALLING ON AN ELEMENTARY RECEIVING AREA HAVING ARBITRARY DIRECTION COSINES AND LIMITED TO THE Z_0 AXIS

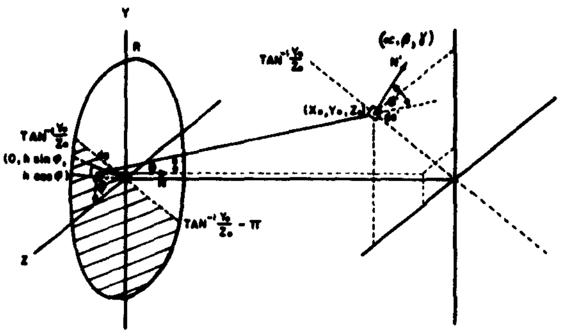

FIG. 15 FLUX FROM A SEMICIRCULAR SOURCE FALLING ON AN ELEMENTARY RECEIVING AREA HAVING ARBITRARY DIRECTION COSINES AND LIMITED TO A PLANE NORMAL TO THE SEMICIRCULAR SOURCE AND THROUGH ITS DIAMETER

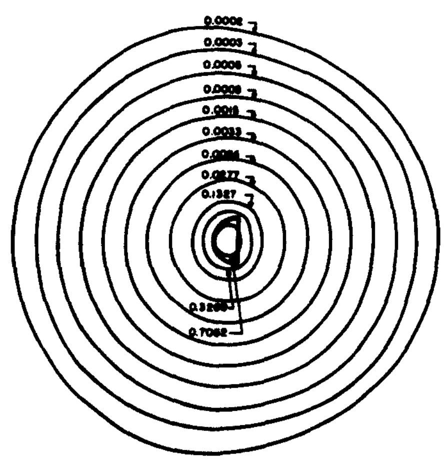

FIG. 16 RADIATION FIELD FOR A SEMICIRCULAR SOURCE FOR $\alpha = 1$ AND $X_0 = R$

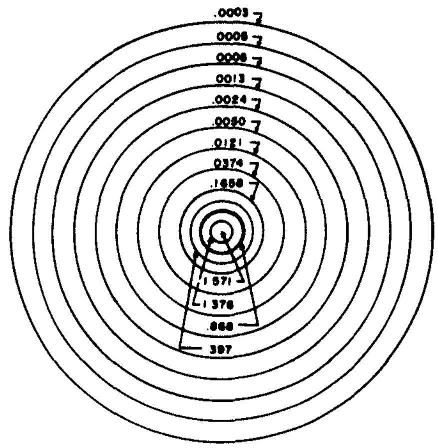

FIG. 17 RADIATION FIELD FOR A CIRCULAR SOURCE FOR $\alpha = 1$ AND $X_0 = R$

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